Fundamental Journals
International Journal of Fundamental Physical Sciences (IJFPS) ISSN:2231-8186
IJFPS, Vol 4, No 4, pp 136-140, Dec , 2014 **R. S.Torogh**
DOI:10.14331/ijfps.2014.330078
http://dx.doi.org/10.14331/ijfps.2014.330078

Existence of Positive Solutions For a Class of Multi-Point P-Laplacian Nonlinear System

R. Sahandi Torogh

Department of Mathematics, Varamin-Pishva Branch, Islamic Azad University, Varamin, Iran

E-mail: sahandi@iauvaramin.ac.ir

(*Received Nov 2014; Published Dec 2014*)

ABSTRACT

In this paper the existence of positive solutions for a class of p-Laplacian boundary value system is studied. In recent years, boundary value problem have received a lot of attention. The fixed point theorems in cones is our main tools to prove the existence of solutions. I provide sufficient conditions under which these systems has positive solutions. I establish some propositions to prove the existence of positive solutions for these equation.

Keywords: Fixed point theorem; p-Laplacian operator; positive solutions

DOI:10.14331/ijfps.2014.330078

INTRODUCTION

In this paper, we prove the existence of positive solutions for the following system,

$$\begin{cases}\left(\emptyset_p(u')\right)' + f(t,u,v) = 0\\ \left(\emptyset_p(v')\right)' + g(t,u,v) = 0\end{cases} \tag{1}$$

$$\begin{cases}u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i) = 0\\ u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i) = 0\end{cases} \quad \begin{cases}v(0) = \sum_{i=1}^{n} \gamma_i v(\xi_i) = 0\\ v(1) = \sum_{i=1}^{n} \partial_i v(\xi_i) = 0\end{cases}$$

Where

$$\phi_p(s) = |s|^{p-2}s, \qquad p > 1, \quad \phi_q = (\phi_p)^{-1},$$

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad \xi_i \epsilon (0,1)$$

with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$.
H_1) $\alpha_i, \beta_i, \gamma_i, \partial_i \epsilon [0,+\infty)$ and

$$0 < \sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \beta_i, \sum_{i=1}^{n} \gamma_i, \sum_{i=1}^{n} \partial_i < 1,$$

$$f, g \in C\left([0,1] \times [0,+\infty) \times [0,+\infty), (-\infty,+\infty)\right)$$

$$f(t,0,0),\ g(t,0,0) \geq 0\ , f(t,0,0),\ g(t,0,0) \neq 0\ .$$

In recent years, BVP have received a lot of attention. There are many papers concerned with the p-Laplacian equations. For example (Liang & Zhang, 2009; Pang, Lian, & Ge, 2007; Sun & Ge, 2007) have studied the existence of positive solutions for some boundary value problems. (Harjani, López, & Sadarangani, 2011) have studied the fixed point theorems in metric spaces. The existence of multiple positive solutions to the boundary value problem was studied by (Ji, Feng, & Ge, 2008).

$$\begin{cases}\left(\emptyset_p(u')\right)' + q(t)f(t,u) = 0,\ t \in (0,1)\\ u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i), u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i)\end{cases}$$

DEFINITIONS

Definition (1) Let $(X, \|.\|)$ be a real Banach space and a non-empty, closed, convex C subset of X is called a Cone of X, If it satisfies the following conditions: i) If $x \in C$ and λ≥ 0 implies that $\lambda x \in C$. ii) If $x \in C$ and $-x \in C$ implies that $x = 0$. Every cone C subset of X includes an ordering in X which is given by $x \leq y$ if and only if $y - x \in C$.

Definition (2) A map $\psi: P \to [0,+\infty)$ is called nonnegative continuous concave functional provided ψ is nonnegative, continuous and satisfies $\psi(tx + (1-t)y) \geq$

$t\psi(x)+(1-t)\,\psi(y)$ for all $x,y\in P$ and $t\in[0,1]$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of X $\beta: P\to[0,+\infty)$ is continuous and $\beta(tx+(1-t)y)\geq t\beta(x)+(1-t)\,\beta(y)$ for all $x,y\in P$ and $t\in[0,1]$.

Definition (3). An operator is called completely continuous if it be continuous and maps bounded sets into pre-compact sets. The main tool of this paper is the following lemma,

Lemma (4) (Lan, 2001) Let B be an open bounded subset of X with $B_k=B\cap K\neq\emptyset$. Assume that $T:\overline{B_k}\to K$ is completely continuous operator such that $Tx\neq x$ for $x\in\partial B_k$, then the following results hold:

i) If $\|Tx\|\leq\|x\|$ for $x\in\partial\Omega_r$ then $i_k(T,B_k)=1$

ii) If there exists $a\in K\backslash\{0\}$ such that $x\neq Tx+\lambda a$ for $u\in\partial B_k$ then $i_k(T,B_k)=0$

iii) Let U be open in X such that $\overline{U}\subset B_k$. If $i_k(T,B_k)=1$ and $i_k(T,U_k)=0$, then T has a fixed point in $B_k\backslash\overline{U_k}$. The same result holds if $i_k(T,B_k)=0$ and $i_k(T,U_k)$ =1.

PRELIMINARIES AND LEMMAS

Let $E=c[0,1]\times c[0,1]$. Then E is a Banach space with the norm $\|(u,v)\|=\|u\|+\|v\|$ where $\|u\|=\max_{0\leq t\leq 1}|u(t)|$. Define the cone K subset of E by

$$K=\{(u,v)\in E\,|\,u,v\ are\ \ concaves\ \ on\ [0,1], u(t)\geq 0, v(t)\geq t\,\}$$

In the rest of this paper we assume that H_2 , H_1 hold.

Lemma (5) (Ji et al., 2008) Suppose $h\big(t,m(t),n(t)\big)>0$, for $m(t)\geq 0, n(t)\geq 0$, then for $m,n\in C^+[0,1]$, the problem

$$\begin{cases}\left(\phi_p(u')\right)'+h(t,m(t),n(t))=0, t\in(0,1)\\ u(0)=\sum_{i=1}^n\alpha_i u(\xi_i), u(1)=\sum_{i=1}^n\beta_i u(\xi_i)\end{cases}\tag{2}$$

Has a solution

$$\begin{aligned}u(t)=\frac{\sum_{i=1}^n\alpha_i}{1-\sum_{i=1}^n\alpha_i}\int_0^{\xi_i}\phi_q(M_{m,n}-\int_0^s h\big(r,m(r),n(r)\big)dr)ds\\ +\int_0^t\phi_q\,(M_{m,n}-\int_0^s h\big(r,m(r),n(r)\big)dr)ds\end{aligned}$$

Where $M_{m,n}$ satisfies

$$\begin{aligned}\frac{\sum_{i=1}^n\alpha_i\,1-\sum_{i=1}^n\beta_i)}{1-\sum_{i=1}^n\alpha_i}\int_0^{\xi_i}\phi_q(M_{m,n}-\int_0^s h\big(r,m(r),n(r)\big)dr)ds\\ +\int_0^t\phi_q\,(M_{m,n}-h\big(r,m(r),n(r)\big)dr)ds\\ -\sum_0^n\beta_i\int_0^{\xi_i}\phi_q(M_{m,n}\\ -\int_0^s h\big(r,m(r),n(r)\big)dr)ds=0\end{aligned}\tag{3}$$

Then there exists a unique $M_{m,n}\in\left(0,\int_0^1 h\big(t,m(t),n(t)\big)dt\right)$ satisfying (3). This implies that there is a unique Δ such that

$$M_{m,n}=\int_0^{\Delta}h\big(t,m(t),n(t)\big)dt)$$

Proof. We define for $h\big(t,m(t),n(t)\big)>0$

$$\begin{aligned}N_{m,n}(l)=\frac{\sum_{i=1}^n\alpha_i(\,1-\sum_{i=1}^n\beta_i)}{1-\sum_{i=1}^n\alpha_i}\int_0^{\xi_i}\phi_q(l\\ -\int_0^s h\big(r,m(r),n(r)\big)dr)ds+\int_0^1\phi_q\,(l\\ -\int_0^s h\big(r,m(r),n(r)\big)dr)ds\\ -\sum_0^n\beta_i\int_0^{\xi_i}\phi_q(l-\int_0^s h\big(r,m(r),n(r)\big)dr)ds\\ =(\frac{\sum_{i=1}^n\alpha_i\,(1-\sum_{i=1}^n\beta_i)}{1-\sum_{i=1}^n\alpha_i}+1-\sum_{i=1}^n\beta_i)\\ \int_0^{\xi_i}\phi_q(l-\int_0^s h\big(r,m(r),n(r)\big)dr)ds+\int_{\xi_i}^1\phi_q(l\\ -\int_0^s h\big(r,m(r),n(r)\big)dr)ds\end{aligned}$$

We see that $N_{m,n}: R\to R$ is continuous and increasing. Then $N_{m,n}(0)$ <0 and $N_{m,n}(\int_0^1 h\big(r,m(r),n(r)\big)dr)>0$ and there exists a $l\in(0,\int_0^1 h\big(t,m(t),n(t)\big)dt)$ Such that $N_{m,n}(l)=0$. Thus there exists Δ such that $M_{m,n}=\int_0^{\Delta}h\big(r,m(r),n(r)\big)dr$.

Lemma(6) (Ji et al., 2008) Suppose that $h\big(r,m(r),n(r)\big)>0$ for $m,n\in C^+[0,1]$. Then the solution of BVP (2) can also be expressed,

$$\begin{aligned}u(t)=\frac{-\sum_{i=1}^n\beta_i}{1-\sum_{i=1}^n\beta_i}\int_{\xi_i}^1\phi_q(N_{m,n}-\int_0^s h\big(r,m(r),n(r)\big)dr)ds-\\ \int_t^1\phi_q\,(N_{m,n}-\int_0^s h\big(r,m(r),n(r)\big)dr)ds,\end{aligned}\tag{4}$$

Where $N_{m,n}$ satisfies

$$\begin{aligned}\frac{\sum_{i=1}^n\beta_i\,(1-\sum_{i=1}^n\alpha_i)}{1-\sum_{i=1}^n\alpha_i}\int_{\xi_i}^1\phi_q(N_{m,n}\\ -\int_0^s h\big(r,m(r),n(r)\big)dr)ds+\int_0^1\phi_q\,(N_{m,n}\\ -\int_0^s h\big(r,m(r),n(r)\big)dr)ds\\ -\sum_0^n\alpha_i\int_{\xi_i}^1\phi_q(N_{m,n}\\ -\int_0^s h\big(r,m(r),n(r)\big)dr)ds\ =0\end{aligned}\tag{5}$$

Then there exists a unique $N_{m,n} \in (0, \int_0^1 h(t, m(t), n(t))dt)$ satisfying (5). This implies that there is a unique $\Delta' \in (0,1)$ such that $N_{m,n} = \int_0^{\Delta'} h(r, m(r), n(r))dr)$.

Lemma (7) (Ji et al., 2008) If $h(r, m(r), n(r)) > 0$ for $m, n \in C^+[0,1]$. Then the solution u(t) of (2) has the following properties,

i) $u(t)$ is concave on (0,1) and $u(t) \geq 0$,

ii) there exists a unique $t_0 \in (0,1)$ such that $u(0) = \max_{0 \leq t \leq 1} u(t)$, $u'(t_0) = 0$,

iii) $\Delta = \Delta' = t_0$.

Lemma (8) (Wang & Zhang, 2006) Let $u(t) \geq 0, u$ is concave on [0,1], $\eta \in (0, \frac{1}{2})$ then $u(t) \geq \eta \|u\|$, $t \in [\eta, 1-\eta]$. We define $\varphi(t) = min\,\{t, 1-t\}, t \in (0,1)$,

$$\gamma_1 = \frac{\eta\, min\left\{\eta \int_\eta^{\frac{1}{2}} \phi_q\left(\frac{1}{2} - s\right) ds, \eta \int_{\frac{1}{2}}^{1-\eta} \phi_q\left(s - \frac{1}{2}\right) \mathrm{ds}\right\}(1 - \sum_{i=1}^n \alpha_i)}{\int_0^1 \phi_q(1-\mathrm{s})\mathrm{ds}}$$

$$\gamma_2 = \frac{\eta\, min\left\{\eta \int_\eta^{\frac{1}{2}} \phi_q(\left(\frac{1}{2} - s\right) ds, \eta \int_{\frac{1}{2}}^{1-\eta} \phi_q\left(s - \frac{1}{2}\right) \mathrm{ds}\right\}(1 - \sum_{i=1}^n \gamma_i)}{\int_0^1 \phi_q(1-\mathrm{s})\mathrm{ds}}$$

$$k_\rho = \{(u,v) \in K|\ \|(u,v)\| < \rho\}$$

$$k^*{}_\rho = \{(u,v) \in K|\ \rho\varphi(t) < u(t) + v(t) < \rho\}$$

$$\Omega_\rho = \left\{(u,v) \in K \middle| \min_{\eta \leq t \leq 1-\eta} (u(t) + v(t)) < \gamma\rho\right\}$$

$$\Omega_\rho = \left\{(u,v) \in K \middle| \gamma\|(u,v)\| \leq \min_{\eta \leq t \leq 1-\eta} (u(t) + v(t)) < \gamma\rho\right\}$$

Lemma (9) Suppose Ω_ρ has the following properties (Lan, 2001),

i) Ω_ρ is open relative to K,

ii) $k_{\gamma\rho} \subset \Omega_\rho \subset k_\rho$,

iii) $(u,v) \in \partial\Omega_\rho$ if and only if

$$\min_{\eta \leq t \leq 1-\eta} (u(t) + v(t)) = \gamma\rho$$

iv) If $(u,v) \in \partial\Omega_\rho$ then $\gamma\rho \leq u(t) + v(t) \leq \rho$ for $\eta \leq t \leq 1-\eta$.

Now, we define,

$$f_{\gamma_1 r}^r = min\left\{\frac{f(t,u,v)}{\phi_p(r)} \middle| t \in [\eta, 1-\eta], u \in [\gamma_1 r, r]\right\}$$

$$g_{\gamma_2 r}^r = min\left\{\frac{g(t,u,v)}{\phi_p(r)} \middle| t \in [\eta, 1-\eta], u \in [\gamma_2 r, r]\right\}$$

$$f_{\varphi(t)r}^r = max\left\{\frac{f(t,u,v)}{\phi_p(r)} \middle| t \in [0,1], u \in [\varphi(t)r, r]\right\}$$

$$g_{\varphi(t)r}^r = max\left\{\frac{g(t,u,v)}{\phi_p(r)} \middle| t \in [0,1], u \in [\varphi(t)r, r]\right\}$$

$$f_0^r = max\left\{\frac{f(t,u,v)}{\phi_p(r)} \middle| t \in [0,1], u \in [0,r]\right\}$$

$$g_0^r = max\left\{\frac{g(t,u,v)}{\phi_p(r)} \middle| t \in [0,1], u \in [0,r]\right\}$$

$$f^\alpha = \lim_{u \to \alpha} max\left\{\frac{f(t,u,v)}{\mathrm{u}^{p-1}} \middle| (t,v) \in [0,1] \times R^+\right\}$$

$$f^\alpha = \lim_{u \to \alpha} max\left\{\frac{f(t,u,v)}{u^{p-1}} \middle| (t,v) \in [0,1] \times R^+\right\}$$

$$g^\alpha = \lim_{v \to \alpha} max\left\{\frac{g(t,u,v)}{v^{p-1}} \middle| (t,u) \in [0,1] \times R^+\right\}$$

$$g^\alpha = \lim_{v \to \alpha} max\left\{\frac{g(t,u,v)}{v^{p-1}} \middle| (t,u) \in [0,1] \times R^+\right\}$$

$$f_\alpha = \lim_{u \to \alpha} min\ \left\{\frac{f(t,u,v)}{\mathrm{u}^{p-1}} \middle| (t,v) \in [0,1] \times R^+\right\}$$

$$g_\alpha = \lim_{v \to \alpha} min\ \left\{\frac{g(t,u,v)}{\mathrm{v}^{p-1}} \middle| (t,u) \in [0,1] \times R^+\right\}$$

MAIN RESULT

Theorem (10) Suppose that H_1 hold and f, g satisfy the following conditions:

H_2) There exist $r_1, r_2, r_3 \in (0, +\infty)$ and $2r_1 < \gamma r_2 < r_2 < 2r_3$ such that

i) $f(t,u,v) > 0, g(t,u,v) > 0, t \in [0,1], u, v \in [r_1\varphi(t), \infty)$

ii) $f_{\varphi(t)r_1}^{r_1} < \phi_p(m), f_{\gamma r_2}^{r_2} > \phi_p(M\gamma)$,

$$g_{\varphi(t)r_1}^{r_1} < \phi_p(m), g_{\gamma r_2}^{r_2} > \phi_p(M\gamma)$$

$$f_{\varphi(t)r_3}^{r_3} \leq \phi_p(m), g_{\varphi(t)r_3}^{r_3} \leq \phi_p(m)$$

H_3)There exists $r_1, r_2, r_3 \in (0, +\infty)$ and $r_1 < r_2 < r_2 < \gamma r_3$ such that

iii)$f(t,u,v) > 0, g(t,u,v) > 0, t \in [0,1]$,

$$u, v \in [min\{\gamma r_1, r_2\varphi(t)\}, \infty);$$

iv) $f_{\gamma r_1}^{r_1} > \phi_p(M\gamma), f_{r_2\varphi(t)}^{r_2} < \phi_p(m)$, $f_{\gamma r_3}^{r_3} \geq \phi_p(M\gamma)$, $g_{\gamma r_1}^{r_1} > \phi_p(M\gamma)$, $g_{\varphi(t)r_2}^{r_2} < \phi_{p_2}(m)$, $g_{\gamma r_3}^{r_3} \geq \phi_p(M\gamma)$,

Then system (1.1) has two positive solutions in K.

Proof. Suppose H_2 holds. We define

$$\bar{f}(t,u,v) = \begin{Bmatrix} f(t,u,v), u \geq r_1\varphi(t)\ , (4.1) \\ f(t, r_1\varphi(t), v(t)), 0 \leq u < r_1\varphi(t) \end{Bmatrix}$$

$$\bar{g}(t,u,v) = \begin{Bmatrix} g(t,u,v), u \geq r_1\varphi(t) \\ g(t,u,r_1\varphi(t)), 0 \leq u < r_1\varphi(t) \end{Bmatrix} \quad (6)$$

So $\bar{f}(t,u,v) \in C([0,1] \times [0,+\infty) \times (0,+\infty), (0,+\infty))$, $\bar{g}(t,u,v) \in C([0,1] \times [0,+\infty) \times (0,+\infty), (0,+\infty))$

We define an operator

$$T: K \to E, T(u,v) = (T_1(u,v), T_2(u,v))$$

$$\left(T_1(u,v)\right)(t) = \left\{\begin{matrix} \frac{\sum_{i=1}^n \alpha_i \int_0^{\xi_i} \phi_q(\int_s^{\sigma_u} \bar{f}(r,u,v))dr)ds}{1-\sum_{i=1}^n \alpha_i} + \int_0^t \phi_q(\int_s^{\sigma_u} \bar{f}(r,u,v))dr)ds\,, 0 \le t \le \sigma_u \\ \frac{\sum_{i=1}^n \beta_i \int_{\xi_i}^1 \phi_q(\int_{\sigma_u}^s \bar{f}(r,u,v))dr)ds}{1-\sum_{i=1}^n \beta_i} + \int_t^1 \phi_{q_1}(\int_{\sigma_u}^s \bar{f}(r,u,v))dr)ds\,, \sigma_u \le t \le 1 \end{matrix}\right\} \tag{7}$$

$$\left(T_2(u,v)\right)(t) = \left\{\begin{matrix} \frac{\sum_{i=1}^n \gamma_i \int_0^{\xi_i} \phi_q(\int_s^{\sigma_v} \bar{g}(r,u,v))dr)ds}{1-\sum_{i=1}^n \alpha_i} + \int_0^t \phi_q(\int_s^{\sigma_v} \bar{g}(r,u,v))dr)ds\,, 0 \le t \le \sigma_v \\ \frac{\sum_{i=1}^n \partial_i \int_{\xi_i}^1 \phi_q(\int_{\sigma_v}^s \bar{g}(r,u,v))dr)ds}{1-\sum_{i=1}^n \beta_i} + \int_t^1 \phi_q(\int_{\sigma_v}^s \bar{g}(r,u,v))dr)ds\,, \sigma_v \le t \le 1 \end{matrix}\right\}$$

Then $T: K \to K$ is completely continuous. We know that for any $(u,v) \in K$:

$$\begin{cases} -(\phi_p\left(\left(T_1(u,v)\right)'\right)'(t) = (\bar{f}(t,u(t),v(t)) \ge 0, \qquad t \in (0,1) \\ -(\phi_p\left(\left(T_2(u,v)\right)'\right)'(t) = (\bar{g}(t,u(t),v(t)) \ge 0 \\ \left(T_1(u,v)\right)(0) = \sum_{i=1}^n \alpha_i\left(T_1(u,v)\right)(\xi_i), \left(T_1(u,v)\right)(1) = \sum_{i=1}^n \beta_i\left(T_1(u,v)\right)(\xi_i) \\ \left(T_2(u,v)\right)(0) = \sum_{i=1}^n \gamma_i\left(T_2(u,v)\right)(\xi_i), \left(T_2(u,v)\right)(1) = \sum_{i=1}^n \partial_i\left(T_2(u,v)\right)(\xi_i), \end{cases} \tag{8}$$

From lemma 7, $\left(T(u,v)\right)(t) = (T_1(u,v), T_2(u,v))(t)$ is concave on [0,1], for $(u,v) \in K$. Then $TK \subset K$. The proof of T is completely continuous is similar to that (Ma, Du, & Ge, 2005). Consider modified problem:

$$\begin{cases} \left(\phi_p(u')\right)' + \bar{f}(t,u,v) = 0 \\ \left(\phi_p(v')\right)' + \bar{g}(t,u,v) = 0 \end{cases}$$

$$\begin{cases} v(0) = \sum_{i=1}^n \gamma_i v(\xi_i) \\ v(1) = \sum_{i=1}^n \partial_i v(\xi_i) \end{cases} \qquad \begin{cases} u(0) = \sum_{i=1}^n \alpha_i u(\xi_i), \\ u(1) = \sum_{i=1}^n \beta_i u(\xi_i), \end{cases}$$

By condition H_2 (ii) we conclude

$$\bar{f}^{r_1}_{\varphi(t)r_1} < \phi_p(m), \bar{f}^{r_2}_{\gamma r_2} > \phi_p(M\gamma),$$

$$\bar{g}^{r_1}_{\varphi(t)r_1} < \phi_p(m), \bar{g}^{r_2}_{\gamma r_2} > \phi_p(M\gamma),$$

$$\bar{f}^{r_3}_{\varphi(t)r_3} \le \phi_p(m), \bar{g}^{r_3}_{\varphi(t)r_3} \le \phi_p(m),$$

By (H_2) we prove that $i_k\left(T, k^*_{2r_1}\right) = 1$. For $(u,v) \in \partial k^*_{2r_1}$, from (7) and $\bar{f}^{r_1}_{\varphi(t)r_1} < \phi_p(m)$, we conclude

$$\left(T_1(u,v)\right)(t) \le \frac{\sum_{i=1}^n \alpha_i \int_0^{\xi_i} \phi_q(\int_s^{\sigma_u} \bar{f}(r,u,v))dr)ds}{1-\sum_{i=1}^n \alpha_i}$$

$$+ \int_0^{\sigma_u} \phi_q(\int_s^{\sigma_u} \bar{f}(r,u,v))dr)ds \le$$

$$\frac{\sum_{i=1}^n \alpha_i \int_0^1 \phi_q(\int_s^1 \bar{f}(r,u,v))dr)ds}{1-\sum_{i=1}^n \alpha_i}$$

$$+ \int_0^1 \phi_q(\int_s^1 \bar{f}(r,u,v))dr)ds$$

$$= \frac{\int_0^1 \phi_q(\int_s^1 \bar{f}(r,u,v))dr)ds}{1-\sum_{i=1}^n \alpha_i}$$

$< \frac{mr_1 \int_0^1 \phi_q(1-s)ds}{1-\sum_{i=1}^n \alpha_i} = r_1 = \frac{\|(u,v)\|}{2}$, so $\|T_1(u,v)\| \le \frac{\|(u,v)\|}{2} = \rho_1$

similarly, we obtain $\|T_2(u,v)\| \le \frac{\|(u,v)\|}{2}$ then

$$\|T(u,v)\| \le \|(u,v)\|, (u,v) \in \partial K^*{}_{2r_1}.$$

From lemma (4) (i) this implies that $i_k\left(T, K^*{}_{2r_1}\right) = 1$. Now, we show that $i_k\left(T, \Omega_{r_2}\right) = 0$ suppose that $a \in \partial k_1$ and $\|a(t)\| = \|(a_1(t), a_2(t))\| = 1, t \in [0,1]$. We prove that

$$(u,v) \ne T(u,v) + \lambda a(t), (u,v) \epsilon \partial\Omega_{r_2}, \lambda \ge 0.$$

Otherwise, there exists $(u_0,v_0) \epsilon \partial\Omega_{r_2}, \lambda_0 \ge 0$ Such that $(u_0,v_0) = T(u_0,v_0) + \lambda_0 a$, i.e.

$$(u_0,v_0) = (T_1(u_0,v_0) + \lambda_0 a_1, T_2(u_0,v_0) + \lambda_0 a_2),$$

So we have $u_0(t) = T_1(u_0,v_0) + \lambda_0 a_1(t)$, (4.3).

We consider two cases:

i) $\sigma_u < \frac{1}{2}$: from (7) and lemma (7), for $t \in [\eta, 1-\eta]$, we have,

$$\|u_0(t)\| = \|T_1(u_0,v_0)(t) + \lambda_0 a_1(t)\| \ge$$

$$\eta\|T_1(u_0,v_0)(t)\| + \lambda_0 \ \ge$$

$$\eta \int_{\sigma_u}^1 \phi_q(\int_{\frac{1}{2}}^s \bar{f}\left(r, u_0(r), v_0(r)\right)dr)ds + \lambda_0$$

$$\ge \eta \int_{\frac{1}{2}}^{1-\eta} \phi_q(\int_{\frac{1}{2}}^s \bar{f}\left(r, u_0(r), v_0(r)\right)dr)ds + \lambda_0 >>$$

$$\eta \int_{\frac{1}{2}}^{1-\eta} \phi_q(\phi_p(r_2)\phi_p(M\gamma))\phi_q(s-\tfrac{1}{2})\lambda_0$$

$$= r_2 M\gamma\eta \int_{\frac{1}{2}}^{1-\eta} \phi_q\left(s-\frac{1}{2}\right)ds + \lambda_0 \ge \gamma r_2 + \lambda_0$$

so we conclude that $\gamma r_2 > \gamma r_2 + \lambda_0$.

ii) $\sigma_u \ge \frac{1}{2}$: from (8) and lemma (8), for $t \in [\eta, 1-\eta]$, we have,

$$\|u_0(t)\| = \|T_1(u_0,v_0)(t) + \lambda_0 a_1(t)\| \geq \eta \|T_1(u_0,v_0)(t)\| + \lambda_0 \quad \geq \eta \int_0^{\sigma_u} \phi_q\big(\int_s^{\sigma_u} \bar{f}(r,u_0(r),v_0(r))dr\big)ds + \lambda_0 \geq$$

$$\eta \int_0^{\frac{1}{2}} \phi_q\big(\int_s^{\frac{1}{2}} \bar{f}(r,u_0(r),v_0(r))dr\big)ds + \lambda_0$$

$$\geq \eta \int_\eta^{\frac{1}{2}} \phi_q\big(\int_s^{\frac{1}{2}} \bar{f}(r,u_0(r),v_0(r))dr\big)ds + \lambda_0$$

$$> r_2 M\gamma\eta \int_\eta^{\frac{1}{2}} \phi_q\big(\frac{1}{2} - s\big)ds + \lambda_0 \geq \gamma r_2 + \lambda_0$$

this implies that $\gamma r_2 > \gamma r_2 + \lambda_0$. From these two cases we conclude $\gamma r_2 > \gamma r_2 + \lambda_0$ which is a contradiction. Then from lemma (4) (ii), we implies that $i_k(T,\Omega_{r_2}) = 0$. Similarly we can prove that $i_k(T,{K^*}_{2r_1}) = 1$. Lemma (4) implies that the problem (8) has three positive solutions such that $(u_1,v_1) \in {K^*}_{2r_1}$, $(u_2,v_2) \in \Omega_{r_2} \setminus \overline{{K^*}_{2r_1}}$, $(u_3,v_3) \in {K^*}_{2r_3}$, It is clear that these solutions belong to $[2r_2\varphi(t),+\infty)$ and satisfy the system (1).

REFERENCES

Harjani, J., López, B., & Sadarangani, K. (2011). Fixed point theorems for weakly C-contractive mappings in ordered metric spaces. *Computers & Mathematics with Applications*, *61*(4), 790-796.

Ji, D., Feng, M., & Ge, W. (2008). Multiple positive solutions for multipoint boundary value problems with sign changing nonlinearity. *Applied Mathematics and Computation*, *196*(2), 511-520.

Lan, K. (2001). Multiple positive solutions of semilinear differential equations with singularities. *Journal of the London Mathematical Society*, *63*(3), 690-704.

Liang, S., & Zhang, J. (2009). The existence of countably many positive solutions for some nonlinear three-point boundary problems on the half-line. *Nonlinear Analysis: Theory, Methods & Applications*, *70*(9), 3127-3139.

Ma, D.-X., Du, Z.-J., & Ge, W.-G. (2005). Existence and iteration of monotone positive solutions for multipoint boundary value problem with p-Laplacian operator. *Computers & Mathematics with Applications*, *50*(5), 729-739.

Pang, H., Lian, H., & Ge, W. (2007). Multiple positive solutions for second-order four-point boundary value problem. *Computers & Mathematics with Applications*, *54*(9), 1267-1275.

Sun, B., & Ge, W. (2007). Existence and iteration of positive solutions for some p-Laplacian boundary value problems. *Nonlinear Analysis: Theory, Methods & Applications*, *67*(6), 1820-1830.

Wang, Z., & Zhang, J. (2006). Positive solutions for one-dimensional p-Laplacian boundary value problems with dependence on the first order derivative. *Journal of Mathematical Analysis and Applications*, *314*(2), 618-630.